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1993 J. Phys. A: Math. Gen. 26 6421

(http://iopscience.iop.org/0305-4470/26/22/037)

J. Phys. A: Math. Gen. 26 (1993) 6421–6430. Printed in the UK

On the fermion quantum white noise calculus

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Received 22 September 1992, in final form 16 August 1993

Abstract. The Fermion interpretation of the white noise framework is considered. As an application, Itô's product formula of Applebaum, Hudson and Parthasarathy is extended to the generalized case.

1. Introduction

The non-commutative (quantum) probability theory has been considerably developed in recent years [1, 2]. In developing their quantum stochastic calculus Hudson and Parthasarathy [3] obtained a quantum (i.e. non-commutative) version of Itô's product formula which was based only on the commutative rules of a free Boson field and Lebesgue integration. The three fundamental integrators are annihilation, creation and number processes which play the role of 'quantum noises' in quantum stochastic evolutions. They are non-commutative extensions of classical Brownian motion and Poisson processes.

On the other hand, the white noise approach initiated by Hida [4] has been proved highly effective to the classical stochastic integration theory [5]. One natural question is: What can one do with it in quantum stochastic calculus?

In fact, some works which connect the quantum probability theory with white noise calculus have appeared. In [6] a quantum white noise calculus leading to an Itô's product formula for more general quantum stochastic measured as a consequence of the Boson commutative relations was developed.

In the present paper, we shall try to give a Fermion interpretation for white noise calculus and extend Itô's product formula of Applebaum, Hudson and Parthasarathy [7] to the more general case.

We briefly recall some notions and notation in white noise calculus [5, 8].

Let $\mathcal{S}(R)$ be the Schwartz's space of rapidly decreasing functions on R and $\mathcal{S}^*(R)$ its dual space. A denotes the self-adjoint extension of the following operator on $H=L^2(R)$:

$$Af(x)=\frac{d^2f}{dx^2}(x)+(1+x^2)f(x) \qquad f\in C_0^\infty(R).$$

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0305-4470/93/226421 + 10 $07.50

This is called the harmonic oscillator operator (see [8, 9] for its physical meaning and further properties). Following Simon [9], we can construct a kind of Sobolev space over R by using operator A. Put

$$e_n(x) =: (-1)^n(\pi^{1/2}2^n n!)^{-1/2}\,e^{x^2/2}\left[\frac{d^n}{dx^n}e^{-x^2}\right] \qquad n \geqslant 0.$$

Then $e_n(x) \in \mathcal{S}(R)$ and we have $Ae_n = 2(n+1)e_n$, $\{e_n,\, n \geqslant 0\}$ is an ONB of $L^2(R)$.

$$\mathcal{S}_p(R) =: \{f \in \mathcal{S}^*(R) \colon \|f\|^2_{2,p} = \|A^p f\|^2_2$$
$$= \sum_{n=0}^{\infty} \langle f, e_n\rangle^2 [2(n+1)]^{2p} < \infty\} \qquad p \geqslant 0.$$

Then $\mathcal{S}_p(R)$ is a Hilbert space and we have

$$\mathcal{S}_q(R) \subset \mathcal{S}_p(R) \qquad \text{for } p < q$$

and

$$\mathcal{S}(R) = \bigcap_{p \in R_+} \mathcal{S}_p(R) \qquad \mathcal{S}^*(R) = \bigcup_{p \in R_+} \mathcal{S}_{-p}(R).$$

Moreover, $\mathcal{S}_{-p}(R)$ is the dual of $\mathcal{S}_p(R)$. The dual pairing $\langle , \rangle$ between $\mathcal{S}_p(R)$ and $\mathcal{S}_{-p}(R)$ is given by

$$\langle \varphi, \psi\rangle = \sum_{n=0}^{\infty} \langle \varphi, e_n\rangle\langle \psi, e_n\rangle.$$

In other words, for $p \geqslant 0$ $\mathcal{S}_p(R)$ is the $L^2(R)$-domain of A^p and

$$\|f\|_{2,p} = \|A^p f\|_2$$

where $\|\cdot\|_2$ is the norm of $L^2(R)$. For $p \leqslant 0$, we can also define the norm $\|\cdot\|_{2,p}$.

For $n \geqslant 2$, we put $\mathcal{S}_p(R^n) =: \{f \in \mathcal{S}^*(R^n) \colon \|f\|_{2,p} < \infty\}$ where

$$\|f\|^2_{2,p} =: \|\Gamma(A)f\|^2_2 = \sum_{k_1 \ldots k_n} \prod_{i=1}^{n} [2(k_i+1)]^{2p} |\langle f, e_{k_1} \otimes \cdots \otimes e_{k_n}\rangle|^2$$

here $\Gamma(A)$ denotes the second quantization operator of A. $\Gamma(A)$ is defined on the dense subset, spanned by $\{f_1 \hat{\otimes} \cdots \hat{\otimes} f_n,\, f_i \in L^2(R)\}$, of the symmetric Fock space of H as follows:

$$\Gamma(A)(f_1 \hat{\otimes} \cdots \hat{\otimes} f_n) =: Af_1 \hat{\otimes} \cdots \hat{\otimes} Af_n$$

(see [5, 7] for second quantization operator $\Gamma(A)$ further properties and physical meanings).

We denote simply by (L^2) the space $L^2(\mathcal{S}^*(R), B(\mathcal{S}^*(R)), \mu)$. Thus for each $\varphi \in (L^2)$ there exists uniquely a sequence $(f^{(n)})$ of functions with $f^{(n)}$ in $L^2(R^n)$ such that φ admits the Itô–Wiener decomposition

$$\varphi = \sum_{n=0}^{\infty} I_n(f^{(n)})$$

where $I_n(f^{(n)})$ is the multiple Wiener integral of $f^{(n)}$ defined by

$$I_n(f^{(n)})=n!\int_{s_1<\cdots<s_n} f^{(n)}(s_1,\ldots,s_n)\,\mathrm{d}B_{s_1}\,\mathrm{d}B_{s_2}\ldots\mathrm{d}B_{s_n}.$$

Here the symbol $\wedge$ means symmetrization. We have

$$\|\varphi\|_2^2=\sum_{n=0}^{\infty} n!\|f^{(n)}\|_2^2.$$

In the sequel we write $\varphi\sim\{f^{(n)}\}$ to specify the sequence $(f^{(n)})$.

Now we can construct Sobolev space over the white noise space. Let $p\geqslant 0$ set

$$(\mathcal{S})_p=\{\varphi\in(L^2):\varphi\sim(f^{(n)}),\sum_{n=0}^{\infty} n!\|f^{(n)}\|_{2,p}^2<\infty\}.$$

Then $(\mathcal{S})_p$ is a Hilbert space with the norm $\|\cdot\|_{2,p}$ given by

$$\|\varphi\|_{2,p}^2=\sum_{n=0}^{\infty} n!\|f^{(n)}\|_{2,p}^2.$$

For each $p>0$, we denote by $(\mathcal{S})_{-p}$ the dual space of $(\mathcal{S})_p$. Here we identify (L^2) with its dual. Each element of $(\mathcal{S})_{-p}$ corresponds uniquely to a sequence $(f^{(n)})$ with $f^{(n)}\in\mathcal{S}_{-p}(R)$ satisfying

$$\|\varphi\|_{2,-p}^2=\sum_{n=0}^{\infty} n!\|f^{(n)}\|_{2,-p}^2<\infty.$$

For $p>0$ we have

$$(\mathcal{S})_p\subset(L^2)\subset(\mathcal{S})_{-p}$$

$\{(\mathcal{S})_p,\ p\in R\}$ are called the Sobolev spaces over the white noise space. Put

$$(\mathcal{S})=\bigcap_{p\in R_+}(\mathcal{S})_p\qquad(\mathcal{S})^*=\bigcup_{p\in R_+}(\mathcal{S})_{-p}.$$

We call the element of $(\mathcal{S})$ (resp. $(\mathcal{S})^*$) Hida test functional (resp. distribution).

The S-transform of functional $F\in(\mathcal{S})^*$ is defined by

$$(SF)(\xi)=\langle\langle F, :e^{\langle\cdot,\xi\rangle}:\rangle\rangle\qquad \xi\in\mathcal{S}(R)$$

where $:e^{\langle\cdot,\xi\rangle}:=\exp\{\langle\cdot,\xi\rangle-1/2\|\xi\|_2^2\}$.

The Hida's differential operator ∂_t $(t\in R)$ is defined as

$$\partial_t\varphi=S^{-1}\left\{\frac{\delta}{\delta\xi(t)}(S\varphi)(\xi)\right\}\qquad \xi\in\mathcal{S}(R)\qquad \varphi\in(\mathcal{S})$$

where $\delta/\delta\xi(t)$ stands for the Fréchet functional derivative [5]. This operator could also be interpreted as a Gâteaux derivative in the direction of δ_t, the Dirac delta function at t. More specifically, let $\varphi\in(\mathcal{S})$ and $x,y\in\mathcal{S}^*(R)$, the Gâteaux derivative of φ at x in the direction y is defined as

$$D_y\varphi(x)=\frac{\mathrm{d}}{\mathrm{d}s}\varphi(x+sy)|_{s=0}.$$

It is known that (cf [5]) for any $y \in \mathcal{S}^*(R)$, D_y is a continuous linear operator on $(\mathcal{S})$ and if $y \in \mathcal{S}(R)$, it can be extended to a continuous linear operator on $(\mathcal{S})^*$. Accordingly, for any $y \in \mathcal{S}^*(R)$, the dual operator D_y^* is a continuous linear operator on $(\mathcal{S})^*$ and if $y \in \mathcal{S}(R)$, its restriction is a continuous linear operator on $(\mathcal{S})$. For the special choice $y = \delta_t$, we have

$$\partial_t = D_{\delta_t}$$

(see [5] for further details about white noise calculus).

In [6] a quantum white noise calculus leading to Itô's product formula for more general quantum stochastic measure as a consequence of the Boson commutative relations was developed which, in its simplest form, uses as annihilation, creation and number processes integrator, the Boson field operators

$$A(t) = W_{0,1}(t) =: \int_{-\infty}^{t} \partial_s \, ds$$

$$A^*(t) = W_{1,0}(t) =: \int_{-\infty}^{t} \partial_s^* \, ds$$

$$N(t) = W_{1,1}(t) =: \int_{-\infty}^{t} \partial_s^* \cdot \partial_s \, ds$$

where ∂_s^* denote the dual of ∂_s.

Also note that

$$Q(t) =: A^*(t) + A(t) \qquad t \in R$$

is defined as a Boson quantum Brownian motion. It is reasonable to call the generalized quantum process

$$X(t) =: \partial_t^* + \partial_t \qquad t \in R$$

a Boson Gaussian white noise. It is also remarkable that process

$$N^\lambda(t) =: N(t) + \sqrt{\lambda} Q(t) + \lambda t$$

is a Quantum Poisson process with parameter λ, so a quantum process

$$n^\lambda(t) =: \partial_t^* \cdot \partial_t + \sqrt{\lambda} X(t) + \lambda I$$

could be reasonably interpreted as a quantum Poisson white noise. Itô's product formula can be summarized by the multiplication rules for stochastic differential:

$$dA(t) \cdot dA^*(t) = dt$$

$$dA(t) \cdot dN(t) = dA(t)$$

$$dN(t) \cdot dA^*(t) = dA^*(t)$$

$$dN(t) \cdot dN(t) = dN(t)$$

which contain the classical Itô's formula as a special case. Other mutual quadratic variations all vanish.

In this paper we develop the Fermion analogy of this Boson theory under the framework of white noise calculus in which the stochastic integrators are now Fermion fields operators. The Fermion annihilation and creation processes can be realized on the white noise spaces by means of an isomorphism between the Boson and Fermion Fock spaces.

We use the following notational convention. The symbol $\wedge$ is usually used to denote the corresponding Fermion case. We denote by σ_n the group $\{1, 2, \ldots, n\}$ and by $\sigma(\pi)$ the sign of an element $\pi \in \sigma_n$.

2. Fermion white noise calculus

Let $H = L^2(R)$

$$\mathcal{F}(H) =: \bigoplus_{n=0}^{\infty} \mathcal{F}_n(H)$$

is the Fock spaces over H. $\mathcal{F}_0(H) = C$ the complex number field

$$S(H) = \bigoplus_{n=0}^{\infty} S_n(H)$$

denotes its symmetric,

$$A(H) =: \bigoplus_{n=0}^{\infty} A_n(H)$$

denotes the antisymmetric Fock spaces over H (see [2, 5] for further details about Fock space). The essential ideas are as follows: One uses a simple isomorphism

$$\mathcal{A} =: \bigoplus_{n=0}^{\infty} \mathcal{A}_n : S(H) \to A(H)$$

between the antisymmetric and symmetric Fock spaces over H. By means of the following chaos isomorphism

$$U_s : S(H) \to (L^2)$$

one obtains immediately an isomorphism

$$U_A = U_S \cdot \mathcal{A}^{-1} : A(H) \to (L^2)$$

which can be generalized to a Hida distribution. The consequences are that an anticommutative field which is usually given on an antisymmetric Fock space can be transported to $(\mathcal{S})^*$, the Hida distribution space.

Proposition 2.2. Let $A(H)$ denote the antisymmetric Fock space on H. Then there exists an isomorphism

$$\mathcal{A} =: \bigoplus_{n=0}^{\infty} \mathcal{A}_n$$

between $S(H)$ and $A(H)$ where $\mathcal{A}_n$ forms an isomorphism unitary mapping from $S_n(H)$ to $A_n(H)$.

Proof. We define $\Delta_n : R^n \to R$ by

$$\Delta_n(x_1, \ldots, x_n) = \theta(|x_1| - |x_2|) \cdot \theta(|x_2| - |x_3|) \cdots \theta(|x_{n-1}| - |x_n|)$$

where θ: $R \to R$ is given by

$$\theta(t)=\begin{cases}0 & t\leq 0\\ 1 & t>0.\end{cases}$$

It is obvious that

$$\sum_{\pi\in\sigma(n)} \Delta_n(\pi x) \overset{a\cdot e}{=} 1$$

where $\pi x=(x_{\pi(1)} \ldots, x_{\pi(n)})$ and assume

$$\varepsilon_n(x_1, \ldots x_n)=\sum_{\pi\in\sigma(n)} \operatorname{sgn}(\pi)\cdot\Delta_n(\pi x)$$

then $|\varepsilon_n(x_1, \ldots, x_n)|=1$ and is almost everywhere a totally antisymmetric function.

Let $F_n \in S_n(H)$, then $\mathscr{A}_n$: $S_n(H)\to A_n(H)$ defined as

$$\mathscr{A}_n F_n(x_1, \ldots, x_n)=\varepsilon_n(x_1, \ldots, x_n)\cdot F_n(x_1, \ldots, x_n)$$

forms an isomorphism unitary mapping from $S_n(H)$ to $A_n(H)$.

Let

$$\mathscr{A}=: \bigoplus_{n=1}^{\infty} \mathscr{A}_n$$

then one obtains immediately an isomorphism unitary mapping

$$\mathscr{A}: S(H)\to A(H)$$

$$\bigoplus_{n=1}^{\infty} F_n \to \bigoplus_{n=1}^{\infty} \mathscr{A}_n F_n \qquad F_n \in S_n(H).$$

Q.E.D.

Remark. A similar Boson–Fermion correspondence was established in [10]. The readers are also recommended to consult [1] for more details.

Now we can construct the so-called Fermion interpretation. Using proposition 2.1, for any $\varphi\sim\{f^{(n)}\}$, we define

$$\hat{\varphi}=\mathscr{A}\varphi\sim\{\mathscr{A}_n f^{(n)}\}=\hat{f}^{(n)}\}$$

and denote

$$(S)_p=:\{\hat{\varphi}\colon \varphi\sim\{f^{(n)}\}, \sum_{n=0}^{\infty} n!\,\|\hat{f}^{(n)}\|^2_{2,p}\,\infty\} \qquad p\geqslant 0, \qquad \|\hat{\varphi}\|^2_{2,p}=:\sum_{n=0}^{\infty} n!\,\|\hat{f}^{(n)}\|^2_{2,p}$$

where $\|\hat{f}^{(n)}\|^2_{2,p}=\|\Gamma(A)^p\hat{f}^{(n)}\|^2_2$ defined on the dense subset, spanned by $\{\hat{f}^{(n)}$: $\hat{f}^{(n)}=f_1\wedge f_2\wedge\cdots\wedge f_n$, $f_i\in L^2(R)\}$, of antisymmetric Fock space on $H=L^2(R)$ as follows: For

$$\hat{f}^{(n)}=f_1\wedge\cdots\wedge f_n \qquad \Gamma(A)^p\hat{f}^{(n)}=:A^pf_1\wedge\cdots\wedge A^pf_n.$$

For each $p>0$, we denote by $(S)_{-p}$ the dual of $(S)_p$. Here we identify (L^2) with its dual space. Let $\mathscr{A}^*$ denote the dual operator of $\mathscr{A}$, each element of $(S)_{-p}$ corresponds uniquely to a sequence $\{\hat{f}^{(n)}\}=\{\mathscr{A}_n^* f^{(n)}\}$ with $f^{(n)}\in S_{-p}^{\wedge}(R^n)$ satisfying

$$\|\hat{\varphi}\|_{2,-p}^2=\sum_{n=0}^{\infty} n!\,\|\hat{f}^{(n)}\|_{2,-p}^2$$

For $p>0$, we have

$$(S)\subset(S)_p\subset(L^2)\subset(S)_{-p}\subset(S)^*$$

$$(S)=\bigcap_{p\in R_+}(S)_p \qquad (S)^*=\bigcup_{p\in R_+}(S)_{-p}.$$

We call the element of (S) (resp. $(S)^*$) the Fermion–Hida test functional (resp, distribution). We denote by $(S)_+$ and $(S)_-$, respectively, the closed spans of the vectors $\{\langle :x^{\otimes n}:, f_1\wedge\cdots\wedge f_n\rangle, f_1,\ldots,f_n\in\mathscr{S}(R)\}$ with n even or odd, respectively, so that thereby (S) is a Z_2-graded rigged Hilbert nuclear space. $(S)^*$ denote their duals. Correspondingly, the operator algebra

$$B((S),(S)^*)=\{T\colon T\colon (S)\to(S)^* \text{ is continuous}\}$$

is Z_2-graded by the rule that is even if $T(S)_\pm\subset(S)_\pm^*$ and odd if $T(S)_\pm\subset(S)_\mp^*$.

3. Fermion Itô product formula

In this section, we shall explicitly realize the Fermion annihilation, creation and the number processes on Hida's white noise functional spaces. As a result, we immediately obtain the corresponding Fermion Itô's product formula by means of the results of [6].

Definition 3.1. A generalized quantum (even or odd) process (GQP) is a pair of densely defined, mutually adjoint families of linear operators $(X(t), X(t)^*;\ t\in R.)$ from (S) into $(S)^*$ such that for any $t\in R$, $X(t)$ (even or odd) with adequate domain contain $C_0=\{\langle :x^{\otimes n}:, f_1\wedge f_2\wedge\cdots\wedge f_n\rangle, f_1,\ldots,f_n\in\mathscr{S}(R).\}$.

Obviously, if we define, for any $\psi\sim\{f^{(n)}\}\in(\mathscr{S})$

$$\mathscr{A}(t)\psi\sim\{\hat{f}^{(n)}_{(-\infty,t)}\otimes f^{(n)}_{[t,+\infty)}\}$$

here assume $\hat{R}_t^n=(-\infty,\ t)\times\ldots\times(-\infty,t)$, $R_t^n=[t,+\infty)\times\ldots\times[t,\ +\infty)$ and $\hat{f}^{(n)}_{(-\infty,t)}$ (resp. $f^{(n)}_{[t,\infty)}$) denote the restriction of $\hat{f}^{(n)}$ (resp. $f^{(n)}$) to $\hat{R}_t^n$ (resp. R_t^n). If $\mathscr{A}^*(t)$ denote the dual of $\mathscr{A}(t)$, $\{\mathscr{A}(t), \mathscr{A}(t)^*, t\in R.\}$ is an even generalized quantum process.

Also, as an important example of GQP, we investigate the Fermion–Hida derivative.

Theorem 3.2. Let ∂_t denote the Hida derivative; we define the Fermion–Hida derivative as follows:

$$D_t=\partial_t\cdot\mathscr{A}(t)$$

$$D_t^*=\mathscr{A}^*(t)\cdot\partial_t^* \qquad t\in R.$$

Then D_t, D_t^* satisfy following anticommutative relations:

$$\{D_t, D_s\}=\{D_t^*, D_s^*\}=0$$

$$\{D_t, D_s^*\}=\delta_s(t)\cdot I$$

where $\{A, B\}=AB+BA$, $\delta_s(t)$ is equal to 1 or 0 whenever $t=s$ or $t\neq s$, respectively.

Proof. Let

$$\psi=\sum_{n=0}^{m}\langle :x^{\otimes n}:, f^{(n)}\rangle\in(\mathcal{S}), f^{(n)}\in\widehat{\mathcal{S}(R^n)}$$

then we have the following Gâteaux differention

$$D_y\langle :x^{\otimes n}:, f^{(n)}\rangle$$
$$=\frac{\mathrm{d}}{\mathrm{d}t}\langle :(x+ty)^{\otimes}:, f^{(n)}\rangle|_{t=0}$$
$$=n\langle :x^{\otimes n-1}:\hat{\otimes}y, f^{(n)}\rangle$$

Hence we obtain

$$D_t\psi(x)=\sum_{n=0}^{m}\langle :x^{\otimes(n-1)}:, \left[\sum_{j=1}^{n}(-1)^{j-1}\hat{f}^{(n)}_{(-\infty,t)}(\cdot\,,\cdot\underset{j}{t},\cdot)\right]\hat{\otimes}nf^{(n)}_{[t,+\infty)}(t,\cdot\cdot)\rangle\,.$$

Here $\underset{j}{t}$ denotes the jth variable location of $\hat{f}^{(n)}$.

Thus we have, without loss of generality supposing $s\leqslant t$

$$D_tD_s\psi(x)=\sum_{n=0}^{m}\langle :x^{\otimes(n-2)}:, [n(n-1)f^{(n)}_{(-\infty,s)}(s,t,\cdot\cdot)]$$
$$\hat{\otimes}\left[\sum_{j\neq k}^{n}(-1)^{j+k}\hat{f}^{(n)}_{[s,t)}(\cdot\cdot\underset{j}{s}\cdot\cdot\underset{k}{t}\cdot\cdot)[n(n-1)f^{(n)}_{[t,\cdot\cdot+\infty)}(s,t,\cdot\cdot)]\right]\rangle$$
$$=-\sum_{n=0}^{m}\langle :x^{\otimes(n-2)}:, [n(n-1)f^{(n)}_{(-\infty,s)}(s,t,\cdot\cdot)]$$
$$\hat{\otimes}\left[\sum_{j\neq k}^{n}(-1)^{j+k}\hat{f}^{(n)}_{[s,t)}(\cdot\cdot\underset{j}{t}\cdot\cdot\underset{k}{s}\cdot\cdot)\right]$$
$$\hat{\otimes}[n(n-1)f^{(n)}_{[t,+\infty)}(s,t,\cdot\cdot)]\rangle=-D_sD_t\psi(x)$$

The conclusion that can be obtained is to note that linear spans of all ψ are dense in $(\mathcal{S})$, that is, we have

$$\{D_t, D_s\}=0 \qquad \text{on } (\mathcal{S}).$$

The other conclusion can be easily obtained similarly.

Q.E.D.

Generally speaking, we can define GQP by any Wick polynomials of QWN. But due to the anticommutation of the Fermion–Hida derivative, the most interesting GQPs are representable by polynomials of, at most, second degree. Next we can construct some important Fermion quantum processes which play the role of 'quantum noise' in quantum stochastic evolution.

The Fermion annihilation and creation processes

$$A_F(t)=:\int_{-\infty}^{t}D_s\,\mathrm{d}s \qquad A^*_F(t)=:\int_{-\infty}^{t}D^*_s\,\mathrm{d}s.$$

Note that

$$Q_F(t) =: A_F^*(t) + A_F(t) \qquad t \in R$$

can be regarded as a generalized quantum Fermion Browian motion. It is reasonable to call the GQP

$$X_F(t) =: D_t^* + D_t \qquad t \in R$$

quantum Fermion Gaussian white noise.

The number process

$$N_F(t) =: \int_{-\infty}^{t} D_s^* \cdot D_s \, ds.$$

It is remarkable that the process

$$N_F^\lambda(t) =: N_F(t) + \sqrt{\lambda} Q_F(t) + \lambda t$$

is a quantum Poisson process with parameter λ. So the QGP

$$N_F^\lambda(t) =: D_t^* \cdot D_t + \sqrt{\lambda} X_F(t) + \lambda I$$

could reasonably be interpreted as a quantum Fermion Poisson white noise.

Remark. A similar 'Fermion Poisson process' was considered in [11].

Some kinds of quantum stochastic differential equation with quantum Fermion Gaussian white noise under the framework of white noise calculus will be considered in separate papers.

Differentiation of these processes yields an operator-valued measure. If we denote $\hat{\partial}_t$ and $\hat{\partial}_t^*$ the canonical annihilation and creation integrator white noise processes in Fermion white noise functional space (S) and $(S)^*$, we have, summarizing section 2 and 3.

Theorem 3.2. (a) There exists a unique unitary mapping $\mathscr{B}$ (resp. $\mathscr{B}^*$) from $(\mathscr{S})$ into itself (resp. $(\mathscr{S})^*$ into itself) such that

$$\mathscr{B} \cdot \hat{\partial}_t \cdot \mathscr{B}^{-1} = D_t$$

$$(\mathscr{B}^*)^{-1} \cdot \hat{\partial}_t^* \cdot \mathscr{B}^* = D_t^*.$$

(b) According to the Itô product formula obtained in [6] and the facts $\mathscr{A}(t) \cdot \mathscr{A}(t)^* = I$, $\mathscr{A}(t) \cdot \mathscr{A}(s) = \mathscr{A}(s) \cdot \mathscr{A}(t)$. $t, s \in R$, we have Itô's product formula:

$$dA_F(t) \cdot dA_F^*(t) = dt$$

$$dA_F(t) \cdot dN_F(t) = dA_F(t)$$

$$dN_F(t) \cdot dA_F^*(t) = dA_F(t)$$

$$dN_F(t) \cdot dN_F(t) = dN_F(t).$$

Other mutual quadratic variations all vanish.

Acknowledgments

The authors thank Professors T Hida and Zhi-Yuan Huang for fruitful discussions, and the referees for their valuable suggestions. The second author is also grateful to Dr Saitô who helped in typing the final manuscript of this paper, and acknowledges the financial support of the Daikô Foundation in Japan.

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